M.Math. IInd year — Midsemestral test 2005
Algebraic number theory — B.Sury
Answer any six INCLUDING THE 9TH

Q 1. If the ring of integers of a number field is a UFD, prove that it must be a PID.

Q 2. Let I be a nonzero ideal in a Dedekind domain and let $a \in I$ be an arbitrary nonzero element. Show that there exists $b \in I$ so that $I = (a, b)$.

Q 3. Let A be a Dedekind domain, K its quotient field and L, a finite separable extension of K. Prove that the integral closure B of A in L is a Noetherian.

Q 4. Let L/K be a Galois extension of number fields. If P is a nonzero prime ideal of $\mathcal{O}_K$, show that $\mathrm{Gal}(L/K)$ acts transitively on the set of prime ideals of $\mathcal{O}_L$ lying over P.

Q 5. Consider $K = \mathbf{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity. Show that a prime $p \in \mathbf{Z}$ splits completely in $\mathcal{O}_K$ if, and only if, $p \equiv 1 \mod n$.

Q 6. Consider a number field K with r_1 real, and $2r_2$ nonreal embeddings into $\mathbf{C}$ over $\mathbf{Q}$. Write down a map which shows how $\mathcal{O}_K$ can be regarded as a lattice in $\mathbf{R}^n$ where $n = r_1 + 2r_2$. Further, prove that the volume of a fundamental parallelotope for this lattice is $\frac{\sqrt{|d_K|}}{2^{r_2}}$.

Q 7. Recall that the Minkowski bound for a number field K of degree n over $\mathbf{Q}$ is $\frac{n!}{n^n}(\frac{4}{\pi})^{r_2}\sqrt{|d_K|}$. Use this to prove that the class number of $\mathbf{Q}(\sqrt{-5})$ is 2.

Q 8. For $K = \mathbf{Q}2^{1/3}$, assume $\mathcal{O}_K = \mathbf{Z}[2^{1/3}]$. Compute the ramification indices e_i and residue field degrees f_i of the prime/primes lying over 5.

Q 9. Recall that if K is a number field with $\mathcal{O}_K = \mathbf{Z}[\alpha]$, the discriminant d_K equals $(-1)^{n(n-1)/2} N_{K/\mathbf{Q}} f'(\alpha)$, where f is the minimal polynomial of α. Consider $K = \mathbf{Q}(\alpha)$, where α is the real root of $X^3 - X - 1$. Assuming that $\mathcal{O}_K = \mathbf{Z}[\alpha]$, find d_K and the best integer given by the Minkowski bound.